CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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C. Tollu, N. Behr, V. Dinh, C. Bui,
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CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of this talk is threefold

Free objects and their combinatorics

A bit of category theory: How to construct free objects w.r.t. a functor and two routes to reach the free algebra.

CRT ?

Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients

MRS and apps

MRS factorisation: A local system of coordinates for Hausdorff groups

Below a quick list of the categories of use in combinatorics (k is a given field), morphisms are standard.

1. **St**, the category of sets
2. **Mon**, the category of monoids
3. **CMon**, the category of commutative monoids
4. **Gp**, the category of groups
5. **Ring**, the category of rings
6. **CRing**, the category of commutative rings
7. $\mathbf{Vect_k}$, the category of k-vector spaces
8. $\mathbf{Lie_k}$, the category of k-Lie algebras
9. $\mathbf{AAU_k}$, the category of k-Associative Algebras with Unit
10. $\mathbf{CAAU_k}$, the category of k-Associative and Commutative Algebras with Unit

Useful categories/2

12. $\mathbf{Alg_k}$, the category of k-Algebras (without conditions)
13. $\mathbf{DiffAlg_k}$, the category of k-Associative Differential Algebras with Unit.
14. $\mathbf{CDiffAlg_k}$, the category of k-Associative Commutative Differential Algebras with Unit.
15. **DiffRing**, the category of Differential rings.
16. **CDiffRing**, the category of Commutative Differential Rings.

Remarks. –
i) All of these have a standard forgetful functor to **St**. They usually compose and factor nicely. See also [20].
ii) For $\mathbf{k} = \mathbb{Z}$, one has

$$\mathbf{DiffAlg}_{\mathbb{Z}} = \mathbf{DiffRing} \text{ and } \mathbf{CDiffAlg}_{\mathbb{Z}} = \mathbf{CDiffRing}.$$

The categories **DiffRing**, **CDiffRing**, $\mathbf{DiffAlg_k}$, $\mathbf{CDiffAlg_k}$

1. We begin with $\mathbf{DiffAlg_k}$
   Let $\mathbf{k}$ be a ring $\mathbf{DiffAlg_k}$ is the category of pairs $(\mathcal{A}, \partial)$ where $\mathcal{A} \in \mathbf{AAU_k}$ and $\partial \in Der(\mathcal{A})$. An arrow $f : (\mathcal{A}, \partial_A) \to (\mathcal{B}, \partial_B)$ is an arrow $f \in Hom_{\mathbf{k}}(\mathcal{A}, \mathcal{B})$ such that $f\partial_A = \partial_B f$.
2. For $(\mathcal{A}, \partial_A) \in \mathbf{DiffAlg_k}$, $ker(\partial_A)$ is a $\mathbf{k}$-subalgebra of $\mathcal{A}$ called that of constants of $\mathcal{A}$.

We now describe the free objects

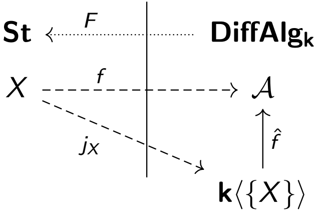
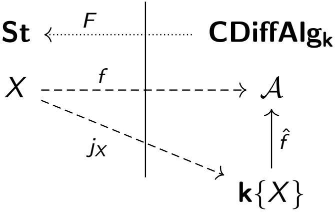


Figure: A solution of the universal problem w.r.t. the natural forgetful functor from $\mathbf{DiffAlg_k}$ to **St**.

Construction of $\mathbf{k}\langle\{X\}\rangle$ and $\mathbf{k}\{X\}$

1. We describe the structure. Let X be an alphabet.
   The free object $\mathbf{k}\langle\{X\}\rangle$ is:
   1. a free algebra $\mathbf{k}\langle X \times \mathbb{N}\rangle$ where, for all $x \in X$, is noted $(x, n) = x^{[n]}$ and, for convenience, $x^{[0]} = x$. This algebra is equipped with the derivation ∂ such that $\partial(x^{[k]}) = x^{[k+1]}$
   2. Existence of ∂ as a derivation is standard (see e.g. [2], Ch I, §2.8 *Extension of derivations*).
   3. The construction is similar to what is to be found in [21], but in the noncommutative realm.
2. We now say a word of the construction in [21]



Construction of $\mathbf{k}\{X\}$

3. Construction of $\mathbf{k}\{X\}$ is very similar to that of $\mathbf{k}\langle\{X\}\rangle$ but
   1. It is devoted to the category $\mathbf{CDiffAlg_k}$ (commutative differential $\mathbf{k}$-algebras)
   2. It uses commutative polynomials i.e. the basic algebra is $\mathbf{k}[X \times \mathbb{N}]$ (and not $\mathbf{k}\langle X \times \mathbb{N}\rangle$) with the same notations ($(x, n) = x^{[n]}$ and $x^{[0]} = x$).
   3. It is the one used for Proposition 2 in Vu's talk (and, in fact, the construction can be done using $\mathbf{k}\{X\}$ with $Y_i^{[j]} = Y_{ij}$ and a suitable ideal).
   4. We recall Proposition 2.

Proposition 2

Let F be a differential field with algebraically closed field of constants C_F and $\mathcal{L}(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + ... + a_1 Y' + a_0 Y = 0$ be defined over F. Then there exists a Picard-Vessiot extension L of F for $\mathcal{L}$, that is unique up to differential F-isomorphism.

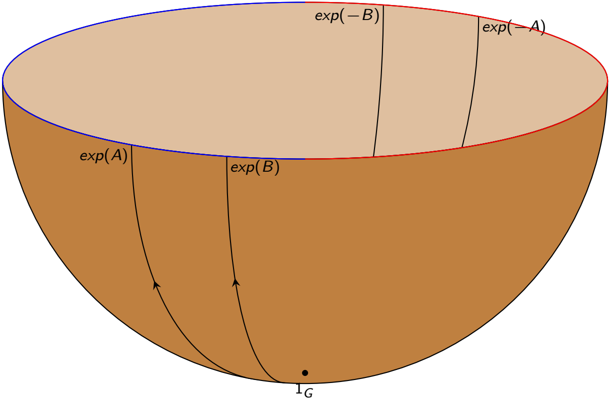
Application: Cartan theorem in Banach algebras (without transversality nor Lipschitz condition)

See `https://mathoverflow.net/questions/356531` for motivation.
Theorem Let $\mathcal{B}$ be a Banach algebra (with unit e) and G be a closed subgroup of $\mathcal{B}^{-1}$ (the group of multiplicative inverses). Let $L(G)$ be the tangent space of G and $m: \ I \to L(G)$ be a continuous function ($I \subset \mathbb{R}$ is an open interval containing $0_{\mathbb{R}}$), then
i) The following system

$$y'(t) = m(t)y(t) \ ; \ y(0) = e$$

admits a unique solution, say $s(t)$.
ii) The trajectory of s is entirely in G (in other words $t \mapsto s(t)$ is a path drawn on G). My questions are the following:
Q1) Is it known? (I expect so, at least of the specialists)
Q2) If yes, is there a sound reference? (not general, but about this very simple and precise property).

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{ш}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

About Magnus expansion and Poincaré-Hausdorff formula/1

Let $(\mathbb{C}\langle\{X\}\rangle, \partial)$ be the differential algebra freely generated by X (a single formal variable). We define a comultiplication Δ by asking that all $X^{[k]}$ be primitive note that Δ commutes with the derivation. Setting, in $\widehat{\mathbb{C}\langle\{X\}\rangle}$, $D = \partial(e^X)e^{-X}$, direct computation shows that D is primitive and hence a Lie series[1], which can therefore be written as a sum of (evaluations of) Dynkin trees.
On the other hand, the formula

$$D = \sum_{k\geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} X^l(\partial X)X^{k-1-l} \cdot \sum_{n\geq 0} \frac{(-X)^n}{n!} \tag{1}$$

suggests that all bidegrees, in $(X, \partial X)$, are of the form $[n, 1]$ and thus, there exists an univariate series $\Phi(Y) = \sum_{n\geq 0} a_n Y^n$ such that $D = \Phi(ad_X)[\partial X]$.

[1]Which would be trivial,if we were in $\mathbb{C}\{X\}$ (i.e. X commutes with ∂X, as there $D = \partial(X)$, but this is not the case within $\mathbb{C}\langle\{X\}\rangle$ as shows the computation (1).

About Magnus expansion and Poincaré-Hausdorff formula/2

Using left and right multiplications by X (resp. noted g, d), we can rewrite (1) as

$$D = \Big(\sum_{k\geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} g^l d^{k-1-l}[\partial X] \Big) e^{-X} \tag{2}$$

but, from the fact that g, d commute, the inner sum $\sum_{l=0}^{k-1} g^l d^{k-1-l}$ is ruled out by the the following identity (in $\mathbb{C}[Y, Z]$, but computed within $\mathbb{C}(Y, Z)$) and

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{\big((Y-Z)+Z\big)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y-Z)^j Z^{k-j}$$

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{\big((Y - Z) + Z\big)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y - Z)^j Z^{k-j} \tag{3}$$

Taking notice that $(g - d) = ad_X$ and pluging (3) into (1), one gets

$$D = \Big(\sum_{k \geq 1} \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (ad_X)^{j-1} d^{k-j} [\partial X] \Big) e^{-X} =$$
$$\frac{1}{ad_X} \Big(\sum_{k \geq 1} \sum_{j=1}^{k} \frac{1}{j!(r-j)!} (ad_X)^j d^{k-j} [\partial X] \Big) e^{-X} = \frac{e^{ad_X} - 1}{ad_X} [X'] \tag{4}$$

which is Poincaré-Hausdorff formula (of course $\dfrac{e^{ad_X} - 1}{ad_X}$ stands for the substitution of ad_X in the formal series corresponding to the entire function $\dfrac{e^z - 1}{z}$).

All here is stated within the same category $\mathcal{C}$.

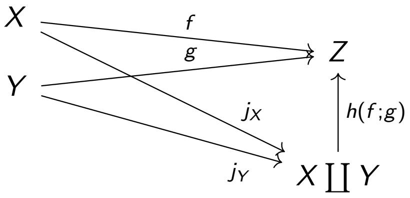


Figure: Coproduct $(j_X, j_Y; X \coprod Y)$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X \coprod Y, Z)\big)\\&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{aligned} \tag{5}$$

Coproducts: Sets

All here is stated within the category **Set**.

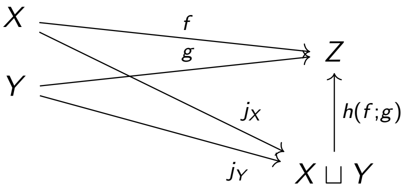


Figure: Coproduct $(j_X, j_Y; X \sqcup Y)$.

$$\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)$$
$$\big(\exists!\, h(f;g) \in Hom(X \sqcup Y, Z)\big)$$
$$\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big) \tag{6}$$

Coproducts: Vector Spaces

All here is stated within the same category $\mathbf{k} - \mathbf{Vect}$.

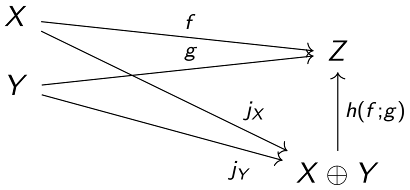


Figure: Coproduct $(j_X, j_Y; X \oplus Y)$ here $h(f;g) = f \oplus g$.

$$\begin{array}{l}\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big) \\ \big(\exists!\, h(f;g) \in Hom(X \oplus Y, Z)\big) \\ \big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{array} \tag{7}$$

Coproducts: $\mathbf{k} - \mathbf{CAAU}$

All here is stated within the same category $\mathbf{k} - \mathbf{CAAU}$.

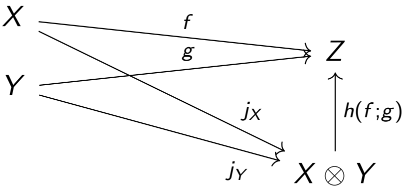


Figure: Coproduct $(j_X, j_Y; X \otimes Y)$ here $h(f;g) = f \otimes g$.

$$\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big) \\ \big(\exists!\, h(f;g) \in Hom(X \otimes Y, Z)\big) \\ \big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big) \tag{8}$$

Coproducts: Augmented $\mathbf{k} - \mathbf{AAU}$

All here is stated within the same category *Augmented* $\mathbf{k} - \mathbf{AAU}$.

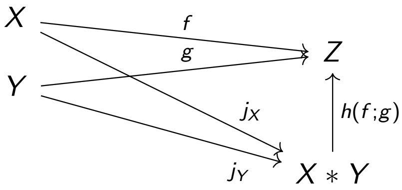


Figure: Coproduct $(j_X, j_Y; X * Y)$ here $h(f;g) = f * g$.

$$\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)$$
$$\big(\exists!\, h(f;g) \in Hom(X * Y, Z)\big)$$
$$\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big) \tag{9}$$

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators

   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra

   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module

    https://ncatlab.org/nlab/show/Hilbert+module

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[2] N. Bourbaki.– *Lie Groups and Lie Algebras, ch 1-3*, Addison-Wesley, ISBN 0-201-00643-X

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[20] Adjuncts in nlab. `https://ncatlab.org/nlab/show/adjunct`

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